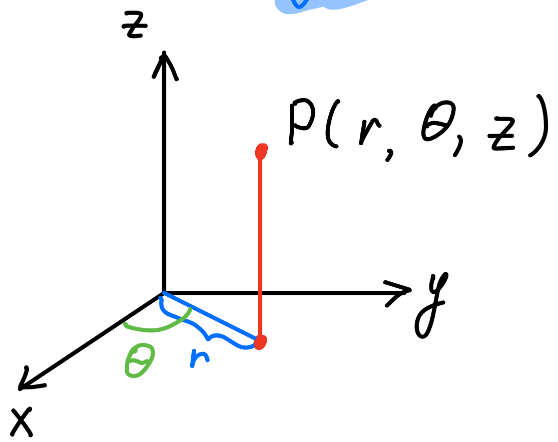
Review

Cylindrical coordinates



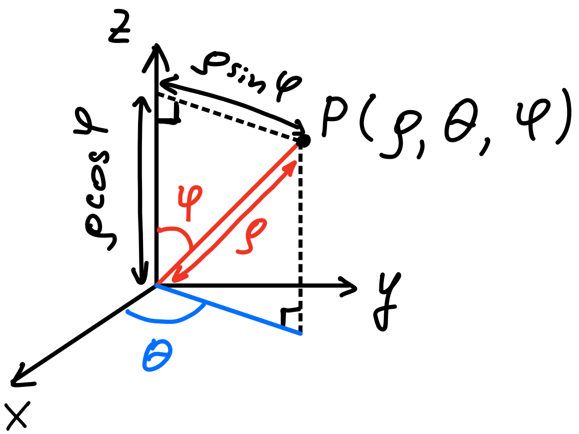
r, θ, z - cylindrical coordinates of a point P

$\underbrace{r, \theta}_{\text{polar coord. of proj. of } P \text{ onto xy-plane}}$

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \Rightarrow x^2+y^2=r^2$$

$$z = z$$

Spherical coordinates:



$\rho = |OP|$

θ - same as in cylindrical

φ - angle between $\vec{OP}$ and z-axis

$\rho \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \varphi \leq \underline{\underline{\pi}}$

$z = \rho\cos\varphi$

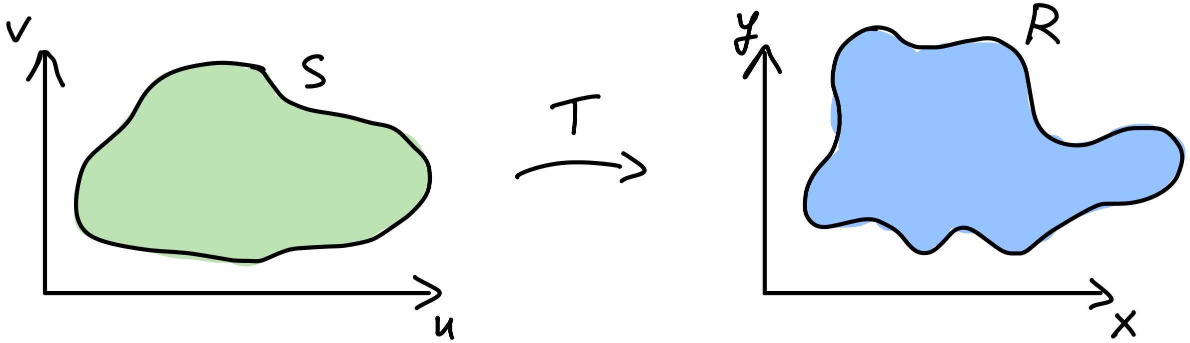
$x = \rho\sin\varphi\cos\theta$

$y = \rho\sin\varphi\sin\theta$

$r = \rho\sin\varphi$

↑ r of cylindrical

<u>Rmk:</u> $\rho^2 = x^2+y^2+z^2$



$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

Jacobian of T:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Change of variables in a double integral:

Suppose T maps S (in uv-plane) to R (in xy-plane) and is differentiable and 1-to-1. Suppose Jacobian $\neq 0$ everywhere inside S.

Then

$$\boxed{\iint_R f(x,y)\,dA = \iint_S f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}$$

Ex: (Polar coordinates)

$T: (r,\theta) \longrightarrow (x,y)$

$x = r\cos\theta$
$y = r\sin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \underbrace{\frac{\partial x}{\partial r}}_{\cos\theta}\underbrace{\frac{\partial y}{\partial \theta}}_{r\cos\theta} - \underbrace{\frac{\partial x}{\partial \theta}}_{-r\sin\theta}\underbrace{\frac{\partial y}{\partial r}}_{\sin\theta} = \underline{\underline{r}} \Rightarrow$$

$$\iint_R f(x,y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)\,\underline{\underline{r}}\,dr\,d\theta$$

Change variables in triple integral:

$$\iiint_R f(x,y,z)\,dV = \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw,$$

where

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Example: $T: (\rho, \theta, \varphi) \mapsto (x,y,z)$, with (Spherical coordinates)

$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho\cos\varphi \end{cases}$$

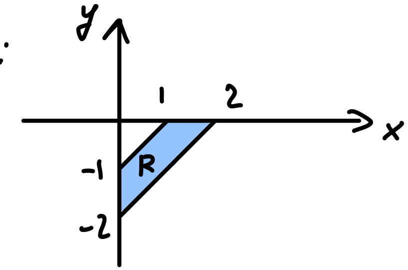
$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix}$$

$$= -\rho^2 \sin\varphi \leq 0 \quad \Longrightarrow$$

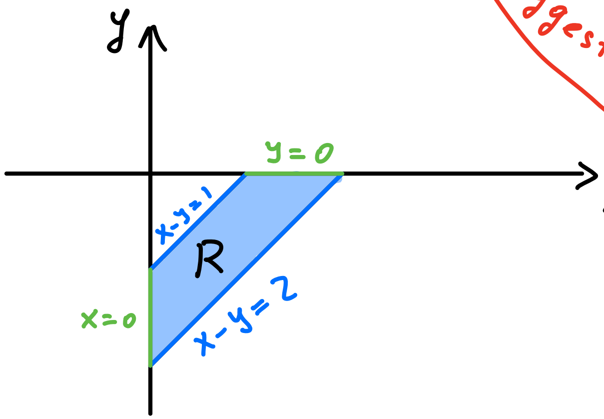
$$\iiint_E f(x,y,z)\,dV = \iiint_S f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi \,d\rho\,d\theta\,d\varphi$$

Review the following example, make sure you can follow all steps!

Ex: Find $I = \iint_R e^{\frac{x+y}{x-y}}\, dA$, where R is a trapezoid:



Sol:

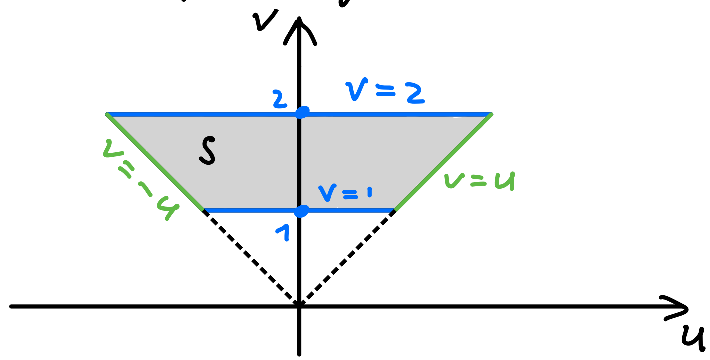


suggested by

Make a change of variables:

$$\begin{cases} x+y=u \\ x-y=v \end{cases} \rightsquigarrow \quad x = \frac{u+v}{2}, \quad y = \frac{u-v}{2}$$

Region S in uv-plane corresponding to R:



$x-y=1 \longrightarrow v=1$

$x-y=2 \longrightarrow v=2$

$y=0 \longrightarrow v=u$

$x=0 \longrightarrow v=-u$

$$S = \left\{ (u,v) : \begin{matrix} 1 \le v \le 2 \\ -v \le u \le v \end{matrix} \right\}$$

Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

So,

$$\boxed{I} = \iint_S e^{\frac{u}{v}} \left|-\tfrac{1}{2}\right| du\, dv = \frac{1}{2} \int_1^2 \underbrace{\int_{-v}^{v} e^{\frac{u}{v}}\, du}_{v e^{\frac{u}{v}} \Big|_{u=-v}^{u=v} = v(e-e^{-1})} dv$$

$$= \frac{1}{2} \int_1^2 (e-e^{-1}) v\, dv = \frac{1}{2}(e-e^{-1}) \frac{v^2}{2} \Big|_{v=1}^{v=2} = \boxed{\frac{3}{4}\left(e - \frac{1}{e}\right)}$$

EXERCISES

a,b) The cylindrical coordinates of the point are given, find rectangular and spherical coordinates of the same point

c,d) The space surface is given by the following equation, describe this surface in cylindrical and spherical coordinates

	Cylindrical (r, θ, z)	Rectangular (x, y, z)	Spherical (ρ, θ, φ)
a)	$(2\sqrt{3}, \pi/3, 2)$		
b)			$(8, \pi/4, \pi/6)$
c)		$x^2 + y^2 + z^2 = 4$	
d)		$x^2 + y^2 = 4$	

True / False:

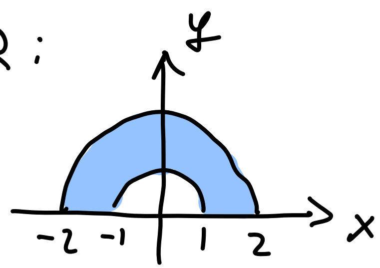
$$\int_1^2 \int_3^4 x^2 e^y \, dy \, dx = \left(\int_1^2 x^2 dx\right)\left(\int_3^4 e^y dy\right) \text{ ?}$$

$$\int_1^2 \int_0^x x^2 e^y \, dy \, dx = \left(\int_1^2 x^2 dx\right)\left(\int_0^x e^y dy\right) \text{ ?}$$

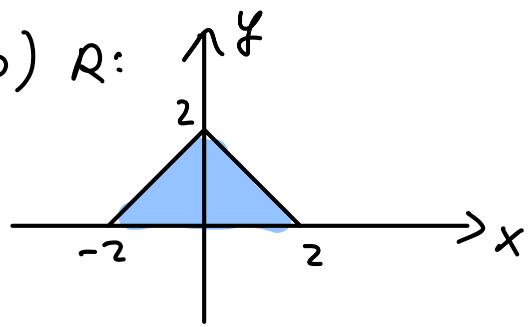
$$\int_0^{2\pi} \int_0^2 \int_r^2 1 \, dz \, dr \, d\theta = \text{the volume enclosed by the cone } z = \sqrt{x^2+y^2} \text{ and the plane } z = 2 \text{ ?}$$

Write $\iint_R f(x,y) \, dA$ as an iterated integral

a) R:



b) R:

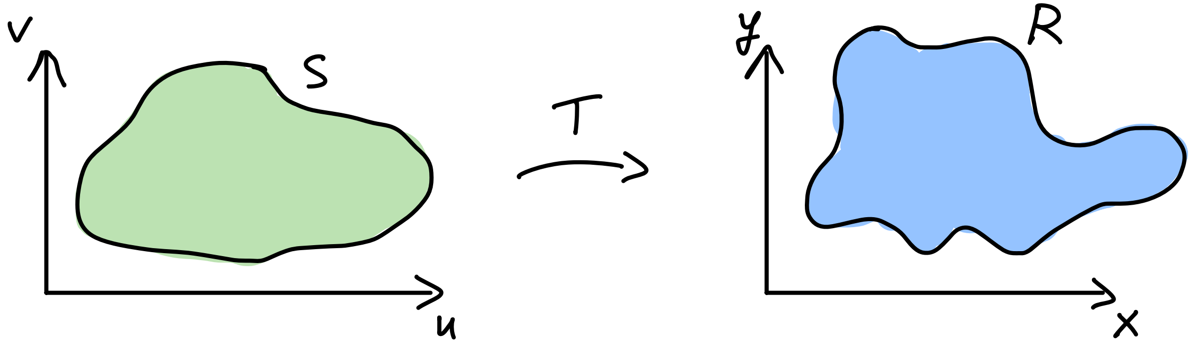


- Evaluate:

a) $$\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^3 + xy^2)\, dy\, dx$$

b) $\iint_D \frac{y}{1+x^2}\, dA$, where D is a region bounded by $y=\sqrt{x}$, $y=0$, $x=1$.

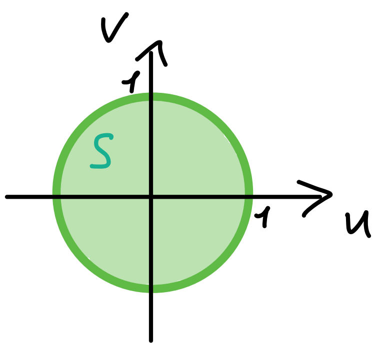
- Use the transformation $x=u^2$, $y=v^2$, $z=w^2$ to find the volume of the region bounded by the surface $\sqrt{x}+\sqrt{y}+\sqrt{z}=1$ and the coordinate planes.



$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

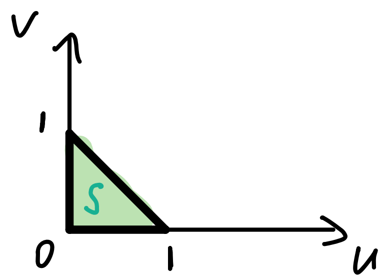
Sketch R and find the Jacobian if

a)



and $\begin{cases} x = u^2 + v^2 \\ y = u^2 \end{cases}$

b)



and $\begin{cases} x = u + v \\ y = u - v \end{cases}$

Solutions

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$x^2+y^2=r^2$$

$$\longleftrightarrow \quad r = \rho\sin\varphi$$

$$x = \rho\sin\varphi\cos\theta$$
$$y = \rho\sin\varphi\sin\theta$$
$$z = \rho\cos\varphi$$
$$\rho^2 = x^2+y^2+z^2$$

Cylindrical (r,θ,z)	Rectangular (x,y,z)	Spherical (ρ,θ,φ)
$(2\sqrt{3}, \pi/3, 2)$	$(\sqrt{3}, 3, 2)$	$(4, \pi/3, \pi/3)$
$(4, \pi/4, 4\sqrt{3})$	$(2\sqrt{2}, 2\sqrt{2}, 4\sqrt{3})$	$(8, \pi/4, \pi/6)$
$r^2+z^2 = 4$	$x^2+y^2+z^2 = 4$	$\rho = 2$
$r = 2$	$x^2+y^2 = 4$	$\rho\sin\varphi = 2$

True / False:

$$\int_1^2\int_3^4 x^2 e^y\,dy\,dx = \int_1^2 x^2\,dx\int_3^4 e^y\,dy$$

TRUE

$$\int_1^2\underbrace{\int_3^4 x^2 e^y\,dy}_{x^2\int_3^4 e^y\,dy \,=\, x^2\left(e^y\big|_3^4\right) \,=\, x^2\left(e^4-e^3\right)}dx = \int_1^2 x^2(e^4-e^3)\,dx = (e^4-e^3)\int_1^2 x^2\,dx = \left(\int_3^4 e^y\,dy\right)\left(\int_1^2 x^2\,dx\right)$$

$$\int_1^2\int_0^x x^2 e^y\,dy\,dx = \int_1^2 x^2\,dx\int_0^x e^y\,dy$$

FALSE

$$\int_1^2\underbrace{\int_0^x x^2 e^y\,dy}_{x^2 e^y\big|_{y=0}^{y=x}}\,dx = \int_1^2 x^2(e^x-1)\,dx = \int_1^2 x^2 e^x\,dx - \int_1^2 x^2\,dx = e^x(x^2-2x+2)\Big|_1^2 - \frac{x^3}{3}\Big|_1^2$$

$$= 2e^2 - e - \frac{7}{3}$$

$$\left(\int_1^2 x^2\,dx\right)\left(\int_0^x e^y\,dy\right) = \frac{7}{3}\cdot\left(e^x-1\right)$$

$$\int_0^{2\pi}\int_0^2\int_r^2 1\,dz\,dr\,d\theta =$$ the volume enclosed by the cone $z=\sqrt{x^2+y^2}$ and the plane $z=2$

FALSE

$$\iiint_E 1\,dV = \int_0^{2\pi}\int_0^2\int_r^2 1\cdot\underline{\underline{r}}\,dz\,dr\,d\theta$$

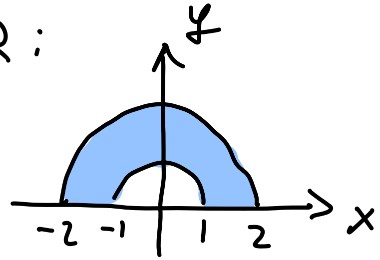
$x = r\cos\theta$
$y = r\sin\theta$
$z = z$
$\sqrt{x^2+y^2} = \sqrt{r^2} = r$

cone: $z = r$

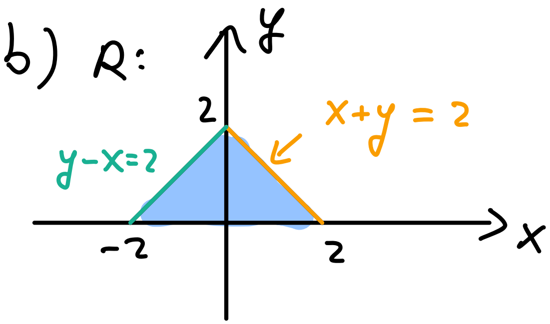
plane: $z = 2$

Write $\iint_R f(x,y)\,dA$ as an iterated integral

a) R:



b) R:



<u>Solution:</u> (a) $R = \left\{ (r,\theta) : \begin{array}{l} 0 \le \theta \le \pi \\ 1 \le r \le 2 \end{array} \right\}$

$$\iint_R f(x,y)\,dA = \int_0^{\pi}\int_1^2 f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

(b) $R = \{ (x,y) : \ 0 \le y \le 2 \ , \ y-2 \le x \le 2-y \}$

$y - x = 2 \iff x = y - 2$

$x + y = 2 \iff x = 2 - y$

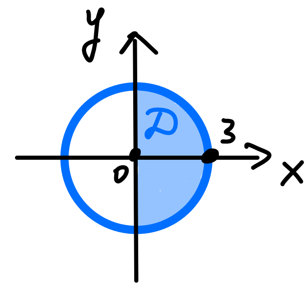
$$\iint_R f(x,y)\,dA = \int_0^2 \int_{y-2}^{2-y} f(x,y)\,dx\,dy$$

- Evaluate:

a) $$\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^3 + xy^2)\,dy\,dx = \iint_{\mathcal{D}} x^3 + xy^2\,dA$$

$$\mathcal{D} = \{(x,y):\ 0 \le x \le 3,\ -\sqrt{9-x^2} \le y \le \sqrt{9-x^2}\}$$

$$\begin{cases} y = \pm\sqrt{9-x^2} \\ x \in [0,3] \end{cases} \Rightarrow \begin{cases} y^2 = 9 - x^2 \\ x \in [0,3] \end{cases} \qquad \begin{cases} x^2 + y^2 = 9 \\ x \in [0,3] \end{cases}$$

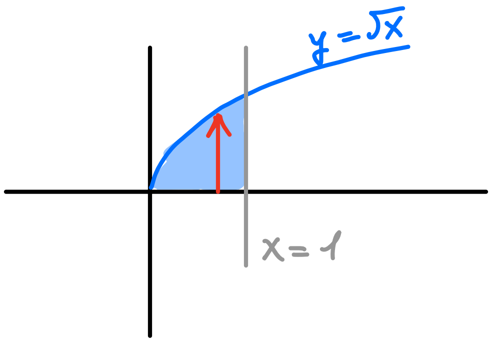


$$\mathcal{D}:\ 0 \le r \le 3,\quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^3 \left((r\cos\theta)^3 + (r\cos\theta)(r\sin\theta)^2 \right) r\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{3^5}{5} \cos\theta\,(\cos^2\theta + \sin^2\theta)\,d\theta$$

$$= \frac{3^5}{5}\left(\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right) = \frac{2\cdot 3^5}{5} = 486/5$$

b) $\iint_{\mathcal{D}} \frac{y}{1+x^2}\,dA$, where $\mathcal{D}$ is a region bounded by $y=\sqrt{x}$, $y=0$, $x=1$.



$$\iint_{\mathcal{D}} \frac{y}{1+x^2}\,dA = \int_0^1 \underbrace{\int_0^{\sqrt{x}} \frac{y}{1+x^2}\,dy}_{\left.\frac{\frac{1}{2}y^2}{1+x^2}\right|_{y=0}^{y=\sqrt{x}}}\,dx = \int_0^1 \frac{1}{2}\,\frac{x^2}{1+x^2}\,dx$$

$$= \frac{1}{2}\int_0^1 \left(1 - \frac{1}{1+x^2}\right) dx$$

$$= \frac{1}{2}\left(x - \arctan(x)\right)\Big|_0^1 = \frac{1-\frac{\pi}{4}}{2}$$

• Use the transformation $x=u^2,\ y=v^2,\ z=w^2$ to find the volume of the region bounded by the surface $\sqrt{x}+\sqrt{y}+\sqrt{z}=1$ and the coordinate planes.

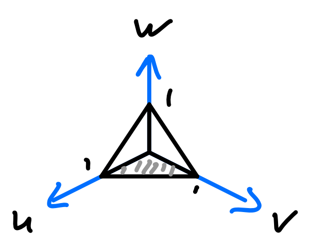
Solution: 1) Jacobian

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 2w \end{vmatrix} = 8uvw \quad \Rightarrow \quad dx\,dy\,dz = 8uvw\,du\,dv\,dw$$

2) Describe the region in (u,v,w)-space:

$$R\left\{\begin{array}{l} \sqrt{x}+\sqrt{y}+\sqrt{z}=1 \\ x,y,z \geq 0 \end{array}\right\} \rightsquigarrow S:\left\{\begin{array}{l} u+v+w=1 \\ u,v,w \geq 0 \end{array}\right\} \leftarrow \text{tetrahedron bounded by } u+v+w=1,\quad u,v,w\geq 0.$$

3) Set up the integral:

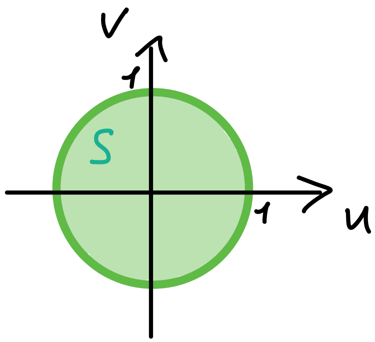


$S:\ 0\leq u\leq 1$
$0\leq v\leq 1-u$
$0\leq w\leq 1-u-w$

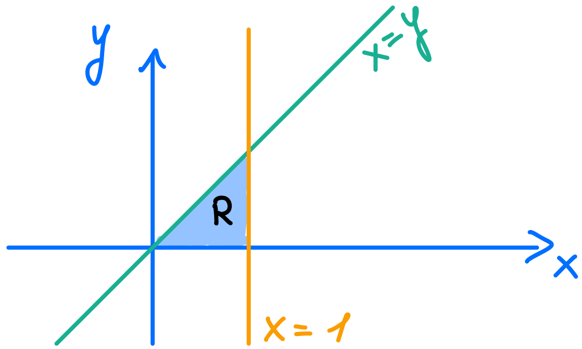
$$V=\iiint_R 1\,dV=\iiint_S 8uvw\,du\,dv\,dw$$

4) Compute the triple integral $\displaystyle\int_0^1\int_0^{1-u}\int_0^{1-u-v} 8uvw\,dw\,dv\,du = \ldots$

a)



and $\begin{cases} x = u^2+v^2 & \geq 0 \\ y = u^2 \end{cases}$



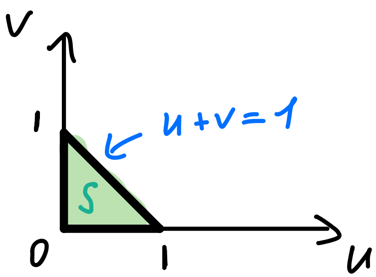
1) $u^2+v^2 \leq 1 \quad \Rightarrow \quad 0 \leq x \leq 1$

$\begin{cases} x = u^2+v^2 \\ y = u^2 \end{cases} \quad \Rightarrow \quad x \geq y$

2)

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = 2u \cdot 0 - 2v \cdot 2u = -4uv$$

b)



and $\begin{cases} x = u+v \\ y = u-v \end{cases}$

2) $\frac{\partial(x,y)}{\partial(u,v)} = 1\cdot(-1) - 1(1) = -2$

1) $u+v=1 \rightsquigarrow x=1$

$u=0,\ v \in [0,1] \to \begin{cases} x = v \\ y = -v \end{cases} \quad \begin{cases} x = -y \\ x \in [0,1] \end{cases}$

$v=0,\ u \in [0,1] \to \begin{cases} x = u \\ y = u \end{cases} \quad \begin{cases} x = y \\ x \in [0,1] \end{cases}$